A Criterion for Isolated Solution Structure and Global Computability for Operator Equations*

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Communicated by Paul. G. Nevai

Received August 22, 1984

DEDICATED TO THE MEMORY OF GÉZA FREUD

This paper considers the description and determination of the fixed points of compact, continuously differentiable mappings T in Banach spaces. In particular, our results describe situations when the fixed point set of T is isolated, and when at least one fixed point can be computed by a globally convergent Newton/continuation algorithm, beginning at zero. The entire framework is motivated by applications to nonlinear elliptic systems for which these properties hold.   © 1986 Academic Press, Inc.

1. INTRODUCTION

Nonlinear elliptic systems constitute an important class of realizations of the operator equation $F(u) = 0$. Gradient systems are reasonably well understood in terms of convex minimization, but systems which are not of gradient type occur frequently enough in the applications to warrant study of those underlying structures which lead to an existence and computational theory, and some understanding of the nature of solutions. Our study (cf. [1]) of the model of the flow of electrons and holes in a semiconductor has suggested such a structure. It is essentially a definition of a solution map, which unfolds component-wise, much as in the definition of iterative methods such as the (nonlinear) Jacobi or Gauss–Seidel iterations, in such a way that the uncoupled equations are individually of gradient type. The fixed points of this map are roots of the system. The solution map T may be shown to be compact and continuous, and invariant on an appropriate closed, convex, bounded set in Hilbert space. The Schauder fixed point theorem guarantees existence, but nothing more. Algorithm determination, for the computation of such fixed points, is a central problem in nonlinear analysis.

We shall present, in this paper, a globally convergent Newton/continuation method, based on the trivial homotopy, for fixed-point mappings T for which the linearized maps do not have positive eigenvalues. This is discussed in a general setting in Section 2. In Section 3, we present an application which is motivated by elliptic systems defined by two dependent variables, one of concentration type and one of potential type, for which the uncouping property mentioned above holds. This does not capture the full generality of the semiconductor model of [1], but does suggest what is possible.

We shall describe now the general results of Section 2. We show here that the existence of a Lipschitz continuous Fréchet derivative T' of T leads to a quadratically convergent local Newton method, provided $T'(v)$ does not have the number one as an eigenvalue, for v in an appropriate compact set K. In particular, mappings whose derivatives have no positive eigenvalues are included. For these mappings, global convergence can be established. Examples are the cyclically-defined mappings described above. This application to cyclically-defined mappings, presented in Section 3, is the genuinely novel contribution of the paper. Our development in the general setting of Section 2 calls upon well-established properties of compactness, of the Fredholm theory, and of the implicit function theorem, and of the author's recent work on Newton's method [2], which makes possible a globally convergent algorithm. Note that the spectral assumption made above, eliminating one as an eigenvalue, leads to the uniform boundedness of the right inverses of $I - T'(v)$ for v in a compact set K, containing the range of T. Now K is a relatively thin set, so that standard arguments employed in the proof on the monodromy theorem (cf. Schwartz [5, pp. 21–23] and Ortega and Rheinboldt [4]) cannot be used to prove uniqueness of solutions without further assumptions. This is, of course, as it should be. It also highlights the severity of assumptions, made at the operator level, on the uniform boundedness, of such linear inversions, on the full ball on which $I - T$ is defined. Such assumptions are tantamount to uniqueness assumptions. This observation does not appear to have been stressed in the literature.

We do not provide proofs of Corollary 2.2 and Proposition 2.4 since the arguments, rather lengthy, are contained in [2]. For the sake of brevity, we assume that the reader has access to this paper, or the report upon which it is based.

2. GLOBAL CONVERGENCE FOR ISOLATED SOLUTIONS

Let X and Z be Banach spaces, with X reflexive and compactly embedded in Z. Let T be a Lipschitz continuously differentiable mapping of

$Cl_Z B_{r_1} \subset Z$ into itself, where we impose the additional conditions that T have range in $Cl_X B_{r_2} \subset X$ and that T have a C^1 extension to an open set $U \supset Cl_Z B_{r_1}$. Here, B_{r_i} are open balls of radius r_i, centered at points z_0 and x_0, in their respective spaces. By standard weak convergence arguments, $Cl_X B_{r_2}$ is compact in Z. It follows that $K = Cl_Z B_{r_1} \cap Cl_X B_{r_2}$ is compact in Z. These properties are assumed to hold throughout this section. The major hypothesis of the section is the following:

(H1) $I - T'(v)$ is injective, for each $v \in K$, as a mapping from Z to Z, hence as a mapping from X to Z.

The following result is standard. A brief sketch of the proof is included to remind the reader of the essential supporting ideas.

PROPOSITION 2.1. *The mapping T has at least one, and at most a finite number, of fixed points in K. Moreover, $I - T'(v)$ is surjective, as a mapping from Z to Z, for each $v \in K$, and $\| [I - T'(v)]^{-1} \|_{Z, Z}$ and $\| [I - T'(v)]^{-1} \|_{Z, X}$ are bounded functions on K.*

Proof.   Fix $v \in K$. The surjectivity of $I - T'(v)$, as a mapping from Z to Z, is immediate from (H1), via the Fredholm alternative (cf. Taylor [6, p. 281]), if we use the compactness of $T'(v)$ (for the latter, cf. Nirenberg [3, p. 58]). We can thus define $[I - T'(v)]^{-1}$ as a bounded linear map from Z to Z, or from Z to X, via the open mapping theorem; $\| [I - T'(v)]^{-1} \|$ is continuous as a function of v in either case, and hence bounded on the compact set K. The Schauder fixed point theorem (cf. Nirenberg [3, p. 33]) guarantees a fixed point of T in the compact, convex subset $K \subset Z$. If there were infinitely many such fixed points, then some fixed point u of K, possibly a boundary point, would be an accumulation point of zeros of the map $F = I - T$. The implicit function theorem (cf. Nirenberg [3, p. 59]), applied to the locally C^1 map

$$H(v, y) = v - T(v) - y \qquad (2.1)$$

at $(v, y) = (u, 0)$ in $Z \times Z$, shows that this cannot happen, since $H_v'(u, 0)$ is invertible from Z to Z, thus assuring that $v = v(y)$, $H(v(y), y) = 0$, is single-valued near $y = 0$. ∎

The preceding proposition allows us to state a local quadratic convergence result. It is based upon recent results of the author (cf. Jerome [2, Section 2]). Precisely the same arguments may be used; $u_k - u_{k-1}$ below must be estimated in both Z and X.

COROLLARY 2.2. *Let M be a positive number satisfying*

$$\| T'(v) - T'(w) \|_{X, Z} \leqslant 2M \| v - w \|_X, \qquad v, w \in K, \qquad (2.2a)$$

and

$$\|[I - T'(v)]^{-1}\|_{Z, Y} \leqslant M, v \in K, \qquad for \ Y = X \ and \ Y = Z. \qquad (2.2b)$$

Suppose ρ is a positive number sufficiently large, as specified in (2.4) to follow. Let u_0 be given, with $u_0 \in B_{\alpha r_1} \cap B_{\alpha r_2}$, $0 \leqslant \alpha < 1$. We assume that h, defined by

$$h := 2(M + M^3) \rho^{-1}, \qquad (2.3)$$

satisfies

$$h \leqslant \frac{1}{2}, \ \frac{1 - \sqrt{1 - 2h}}{2(1 + M^2)} \leqslant (1 - \alpha) \min(r_1, r_2); \qquad (2.4a)$$

we also assume the residual condition

$$\|F(u_0)\|_Z \leqslant \rho^{-1}. \qquad (2.4b)$$

Then, the Newton iterates, defined formally by

$$u_k - u_{k-1} = -[I - T'(u_{k-1})]^{-1} [I - T(u_{k-1})], \qquad k \geqslant 1, \qquad (2.5)$$

are well defined and lie in K; the sequence $\{u_k\}$ converges quadratically to a root $u \in K$ of $F = I - T$, $F(u) = 0$, given by the estimate

$$\|u - u_k\|_X \leqslant \frac{\omega_k}{2(1 + M^2)} \frac{(1 - \sqrt{1 - 2h})^{2^k}}{2^k}, \qquad (2.6)$$

for $k = 1, 2, \ldots$. Here $0 < \omega_{k+1} \leqslant \omega_k \leqslant 1$ for $k \geqslant 1$.

Remark 2.1. The numbers ω_k can be defined recursively (see [2]). The sequence $\{\omega_k\}$ is quadratically convergent to 0 near $h = \frac{1}{2}$; this information is not contained in the usual Kantorovich recurrence relations, and appears to be new to [2]. Hypothesis (H1) is apparently not sufficient to yield a globally convergent algorithm, via continuation from some trivial solution. For example, if we consider the case where B_{r_1} and B_{r_2} are centered at 0 and also consider the trivial homotopy,

$$F(v, \lambda) = v - \lambda T(v), \qquad (2.7)$$

for $v \in K$ and $0 \leqslant \lambda \leqslant 1$, then

$$F(0, 0) = 0 \qquad (2.8)$$

provides an evident solution at the commencement of the homotopy. The

global theory developed in [2] provides for a predictor–corrector Euler–Newton continuation from $u = 0$ provided, for $v \in K$ and $0 \leqslant \lambda \leqslant 1$,

$$\| [F_v'(v, \lambda)]^{-1} \|_{Y, Z} \leqslant M, \qquad \text{for } Y = X \text{ and } Y = Z. \tag{2.9}$$

We shall see shortly that the following hypothesis implies (2.9):

(H2) $T'(v)$ has no positive eigenvalues, as a mapping from Z to Z, for each $v \in K$.

LEMMA 2.3. *Inequality* (2.9) *is implied by* (H2).

Proof. A direct computation gives

$$F_v'(v, \lambda) = v - \lambda T'(v), \qquad \text{for } v \in K. \tag{2.10}$$

We select $0 < \lambda_0 < 1$ such that, for $v \in K$,

$$\lambda_0 \| T'(v) \|_{Y, Z} \leqslant \gamma < 1, \qquad \text{for } Y = X \text{ and } Y = Z. \tag{2.11}$$

This is possible by the smoothness of $T(\cdot)$ and the compactness of K. Clearly, $F_v'(\cdot, \lambda)$ is invertible for $0 \leqslant \lambda \leqslant \lambda_0$, with inverse bound $(1 - \gamma)^{-1}$, as a mapping from Z to Z or from X to Z. For $\lambda_0 \leqslant \lambda \leqslant 1$, the argument of Proposition 2.1 applies to bound the norm of the inverse on the compact set, $K \times [\lambda_0, 1]$. In particular, (2.9) follows from the conjunction of these two arguments. ∎

Remark 2.2. We shall present a result which embodies a global continuation method, which tracks a solution of $F(u) = F(u, 1) = 0$ by beginning at 0, as in (2.8). The method may be described conceptually as following the solution curve $u = u(\lambda)$, in the sense that the approximations lie within an envelope, defined by balls of radius $M\rho^{-1}$ and centered at $u = u(\lambda)$. This envelope has its cross sections in the domain of convergence of Newton's method. The algorithm advances along the mesh points i/N of the homotopy interval $[0, 1]$ by predicting, from the previous corrector, in such a way that the change in the residual is negligible (see (2.14a) and (2.15)). This is followed at each mesh point by one Newton corrector iteration (see (2.14b)). We shall present the result without proof. The details are contained in [2, Sect. 3]. Some simplification results from the fact that the trivial homotopy, (2.7), is employed. In particular, the requisite regularity in (v, λ) required for F is implied by the assumed regularity of T. Also, the predictor appears formally as a damped Newton iteration. We pass now to the result.

PROPOSITION 2.4. *Suppose M is defined by* (2.2a) *and* (2.9), *suppose h*

and $\rho \geqslant 1$ satisfy (2.3) and (2.4a), with $\alpha = \frac{3}{4}$, and suppose $\Delta\lambda$ is selected to satisfy

$$N^{-1} = \Delta\lambda \leqslant (8M\rho)^{-1}. \tag{2.12}$$

Then, predictor-corrector sequences $\{v_i\}_1^N$ and $\{w_i\}_0^{N-1}$ exist in K for which $w_0 = 0$ and

$$\|F(v_i, i/N)\|_Z \leqslant \rho^{-1}, \qquad \|F(w_i, i/N)\|_Z \leqslant (2\rho)^{-1}. \tag{2.13}$$

The sequence $\{v_i\}$ lies in the intersection of the closed balls $Cl_Z B_{3r_1/4}$ and $Cl_X B_{3r_2/4}$, and is defined explicitly by

$$v_i = w_{i-1} + \Delta\lambda [F_v'(w_{i-1}, (i-1)/N)]^{-1} T(w_{i-1}), \tag{2.14a}$$

whereas the corrector sequence $\{w_i\}$ lies in the intersection, $Cl_Z B_{r_1/2} \cap Cl_X B_{r_2/2}$, and is defined by

$$w_i = v_i - [F_v'(v_i, i/N)]^{-1} F(v_i, i/N). \tag{2.14b}$$

Remark. The significance of the result is that v_N provides a starting guess for Newton's method, for which convergence to the desired fixed point is guaranteed. The predictors v_i are defined by an explicit Euler discretization, applied to the differential equation

$$\frac{d}{d\lambda}(F(u(\lambda), \lambda)) = 0, \tag{2.15}$$

on the partition i/N, $i = 0,..., N$.

3. Solution Maps of Cyclic Type

We discuss a class of smooth nonlinear elliptic systems, motivated by the model of [1]. For simplicity, we shall consider a system of two equations, in divergence form, for a pair u, v defined on a bounded, open region Ω in R^n and satisfying specified values on the uniformly Lipschitz boundary $\partial\Omega$ of Ω. The system involves a potential variable u and a concentration variable v, where the relation $u = S(v)$ is defined by the first equation. The system is given by

$$-\Delta u + a(u, v) = q, \qquad \text{in } \Omega, \tag{3.1a}$$

$$-\nabla \cdot (f(S(v) - v)\nabla v) = 0, \qquad \text{in } \Omega, \tag{3.1b}$$

$$u = \bar{u}, \qquad v = \bar{v} \qquad \text{on } \partial\Omega. \tag{3.2}$$

Here, a and f are sufficiently smooth with $a = b_u$, where b is convex in u, f, and f' are positive, and q is given.

In the framework of the previous section, we might wish to select $Z = C(\bar{\Omega})$, and $X = H^r(\Omega)$, for r sufficiently large, say $r > n/2$. The mapping S is assumed smooth on Z, such that, for $\tilde{v}$ to be specified:

(H3) $A(\tilde{v}) := (f'/f)(S(\tilde{v}) - \tilde{v}) \times [I - S'(\tilde{v})]$ is bounded, with extension to a positive-definite, self-adjoint operator on $L^2(\Omega)$, with the same spectrum.

Note that the maps of (H3) involve composition with multipliers assumed bounded. Also, the system is assumed elliptic, with intrinsic maximum principles, in the sense that the mapping T, written formally in terms of

$$-\nabla \cdot (f(S(v) - v) \nabla T(v)) = 0, \qquad \text{in } \Omega, \qquad (3.3\text{a})$$

$$T(v) = \tilde{v}, \qquad \text{on } \partial\Omega, \qquad (3.3\text{b})$$

satisfies:

(H4) T is a compact, smooth map of a compact subset K of Z into itself. K is also assumed convex and contained in X.

The term smooth here means Lipschitz continuous differentiability. The function $\tilde{v}$ of (H3) may be an arbitrary member of K. Note that fixed points v of T define solutions of (3.1), (3.2) in terms of $(S(v), v)$.

Remark. 3.1.   The semiconductor model of [1], in simplified form, is given by (3.1), with $f = \exp$, and $u = S(v)$ rendered by the solution of a boundary-value problem defined by the equation:

$$-\Delta u + \exp(u - v) - \exp(-u) = q. \qquad (3.4)$$

This explains assumption (H3). In conjunction with the way T unfolds in (3.3), it defines a map cyclically. The reader may replace $S(v) - v$, in (3.1b), by an appropriate generalized expression which retains the assumed definiteness.

In addition to (H3) and (H4) we also assume:

$$B(v) := -\nabla \cdot (f(S(v) - v) \nabla), \qquad v \in K,$$

(H5)   are positive-definite, self-adjoint, unbounded operators on $L^2(\Omega)$, with a fixed, common domain D of functions vanishing on $\partial\Omega$. The positive-definite, self-adjoint mappings $(A(v) B(v))^* A(v) B(v)$ are assumed to have compact inverses.

Remark 3.2. The conjunction of (H3) and (H5) yields the conclusion when account is taken of the pointwise positivity of A and B^{-1},

$$(A(v) \circ B(v) \, \phi, \, \phi)_{L^2(\Omega)} \leqslant 0 \Rightarrow \phi = 0 \qquad \text{for } \phi \in L^2(\Omega), \tag{3.5}$$

where $A(v)$ is the operator in (H3), and $B(v)$ is the operator in (H5). The structure for A, B also yields complete orthogonal sequences $\{u_i\}$ and $\{v_i\}$, for which $\{u_i\}$ is orthonormal, and

$$ABu_i = \lambda_i v_i, \qquad \lambda_i > 0, \qquad \text{each } i. \tag{3.6a}$$

This is possible if $\{u_i\}$ is constructed, via the spectral theory, from

$$(AB)^* ABu_i = \lambda_i^2 u_i, \quad \text{all } i, \tag{3.6b}$$

and the sequence $\{v_i\}$ is defined by

$$v_i = \lambda_i^{-1} ABu_i, \qquad \text{all } i. \tag{3.6c}$$

Note that $\lambda_i^2 > 0$ follows from the positive definiteness of A and B. The orthogonality of $\{v_i\}$ follows from (3.6b), and the completeness from the positive definiteness of A and B. In the inner product, the term by term application of AB in the $\{u_i\}$-expansion of ϕ is permitted, but this does not yield (3.5), which requires pointwise positivity.

The major result of this section is the following.

PROPOSITION 3.1. *For each $v \in K$, the Jacobian map $T'(v)$ does not possess real eigenvalues λ, except possibly for $\lambda = 0$.*

Proof. We compute $T'(v)[\phi]$ implicity by differentiating (3.3a) with respect to v. We obtain the equation, for $\phi \in X \cap D_{T'(v)}$

$$-\nabla \cdot (f'(S(v) - v)[S'(v) - I] \phi \nabla T(v)) - \nabla \cdot (f(S(v) - v) \nabla T'(v) \, \phi) = 0. \tag{3.7}$$

Now suppose that

$$T'(v) \, \phi = \lambda \phi, \tag{3.8}$$

for some real number λ, $\lambda \neq 0$. We multiply (3.7) by the function

$$\Psi = \left(\frac{f'}{f}\right)(S(v) - v)[S'(v) - I] \, \phi, \tag{3.9}$$

with homogeneous boundary values, integrate, and apply the divergence theorem, recognizing that ϕ and Ψ vanish on $\partial\Omega$. We observe that the term

$$\frac{1}{2}\int_\Omega f(S(v)-v)\,\nabla T(v)\cdot\nabla\left|\left(\frac{f'}{f}\right)(S(v)-v)[S'(v)-I]\,\phi\right|^2,$$

which results from the first term of (3.7), is zero according to (3.1b). An application of (3.8) to the second term, followed by division by λ, yields

$$(A(v)\circ B(v)\,\phi,\,\phi)_{L^2(\Omega)}=0,$$

for $A(v)$ and $B(v)$ as described in Remark 3.2. It follows that $\phi=0$ and the proof is concluded. ∎

Remark 3.3.   The theory of global convergence thus applies to systems such as (3.1). The theory applies 'a fortiori' to simpler semilinear systems such as

$$0=-\Delta u_i+R_i(u_1,...,u_m),\qquad i=1,...,m,$$

where R_i is a smooth, Lipschitz continuous function depending monotonely upon each u_j, and vanishing at 0. In these applications, it is assumed that 0 is an interior point.

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